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Existence, regularity and local uniqueness of the solutions to the Maxwell–Landau–Lifshitz system in three dimensions ☆

Ivan Cimrák *

Numerical Functional Analysis and Mathematical Modeling Research Group, Department of Mathematical Analysis, Ghent University, Galglaan 2, 9000 Ghent, Belgium

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Abstract

We study a model of ferromagnetism governed by Maxwell's equations coupled with the non-linear Landau–Lifshitz equation of micromagnetism. We are interested in the cases of space-periodic solutions for 3D domains. Obtaining the regularity of the solution $\boldsymbol{m}$ in $\boldsymbol{W}^{2,2}_{\mathrm{per}}(\Omega)$ space and of the solutions $\boldsymbol{E}$, $\boldsymbol{H}$ in $\boldsymbol{W}^{1,2}_{\mathrm{per}}(\Omega)$ space we state the existence theorem. Finally, we prove the local uniqueness of the solutions.

Keywords: Ferromagnetism; Micromagnetics; Spin-chain; Existence; Regularity; Maxwell's equations

1. Introduction

We discuss the existence, uniqueness and regularity of a space-periodic solution to the coupled full Maxwell–Landau–Lifshitz (M-LL) system for 3D domains. The system reads as

$$\partial_t \boldsymbol{m} = -\alpha_1 \boldsymbol{m} \times (\Delta \boldsymbol{m} + \boldsymbol{H}) - \alpha \boldsymbol{m} \times \big(\boldsymbol{m} \times (\Delta \boldsymbol{m} + \boldsymbol{H})\big), \tag{1}$$

$$\partial_t \boldsymbol{E} + \sigma \boldsymbol{E} - \nabla \times \boldsymbol{H} = \boldsymbol{0}, \tag{2}$$

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* Fax: 0032 9 2644987.

E-mail address: ivan.cimrak@ugent.be.

URL: http://cage.ugent.be/~cimo.

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$$\partial_t \boldsymbol{H} + \nabla \times \boldsymbol{E} = -\beta \partial_t \boldsymbol{m}, \tag{3}$$

$$\nabla \cdot \boldsymbol{H} + \beta \nabla \cdot \boldsymbol{m} = 0, \tag{4}$$

$$\nabla \cdot \boldsymbol{E} = 0, \tag{5}$$

in $\Omega \subset \mathbb{R}^3$. Constants α_1, α, β and σ are of physical origin, $\alpha > 0$, $\sigma \geqslant 0$. We denote the magnetic field, the electric field and the magnetization by $\boldsymbol{H}$, $\boldsymbol{E}$ and $\boldsymbol{m}$, respectively.

We are interested in the case when the solutions are space-periodic functions. This model arises from various physical and engineering applications such as electromagnetic wave propagation or antennas. It can be used also for the study of electromagnetic behavior of the materials with a periodical structure.

The function spaces that correspond to space-periodic functions will be denoted by a subscript "per," e.g., $\boldsymbol{W}^{1,2}_{\mathrm{per}}(\Omega)$.

After these considerations, the working domain can be handled as a cube $\Omega = (0, D)^3$ with periodic boundary conditions in each space direction, i.e., for $\boldsymbol{x} \in \Omega \subset \mathbb{R}^3$, and $t \geqslant 0$,

$$\begin{aligned} &\boldsymbol{m}(\boldsymbol{x} + D\boldsymbol{e}_i, t) = \boldsymbol{m}(\boldsymbol{x}, t), \qquad \boldsymbol{H}(\boldsymbol{x} + D\boldsymbol{e}_i, t) = \boldsymbol{H}(\boldsymbol{x}, t), \\ &\boldsymbol{E}(\boldsymbol{x} + D\boldsymbol{e}_i, t) = \boldsymbol{E}(\boldsymbol{x}, t), \end{aligned} \tag{6}$$

where $\boldsymbol{x} + D\boldsymbol{e}_i = (x_1, \ldots, x_i + D, \ldots, x_3)$, $i = 1, 2, 3$ and $D > 0$.

This kind of domain equipped with periodic boundary conditions can be in mathematical analysis considered as a domain without boundary. Such kinds of domains for similar physical problems were studied, e.g., by Prohl in [15], Guo and Hong in [11], Guo and Su in [12], and other authors.

For the cases with non-periodic domain there arise difficulties in some steps of the proofs. One is not always able to get rid of boundary terms while performing integration by parts. Therefore for simplicity we restrict ourselves to the periodic types of the domain.

The initial conditions read as

$$\boldsymbol{m}(x, 0) = \boldsymbol{m}_0(x), \quad \boldsymbol{H}(x, 0) = \boldsymbol{H}_0(x), \quad \boldsymbol{E}(x, 0) = \boldsymbol{E}_0(x), \quad x \in \Omega. \tag{7}$$

A crucial observation is, that $|\boldsymbol{m}| = 1$ for almost all $t \in \langle 0, \infty)$ provided that the solution to (1)–(5) is sufficiently smooth. This comes from a scalar multiplication of (1) with $\boldsymbol{m}$. Then Eq. (1) is equivalent to

$$\partial_t \boldsymbol{m} - \alpha \Delta \boldsymbol{m} = \alpha |\nabla \boldsymbol{m}|^2 \boldsymbol{m} - \alpha_1 \boldsymbol{m} \times \Delta \boldsymbol{m} - \alpha_1 \boldsymbol{m} \times \boldsymbol{H} - \alpha \boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{H}). \tag{8}$$

The transformation of Eq. (1) to Eq. (8) is a classical approach used, for example, in [4,10,15]. In sequel we will work with (8) rather then with (1), because (8) contains the term with Laplace operator separated from the rest.

Problem P. We will refer to the system of Eqs. (8), (2)–(5) equipped with boundary conditions (6) and initial conditions (7) as to Problem P.

2. Overview of known results

For the modeling ferromagnetic phenomena on micro-scales the so called Landau–Lifshitz (LL) equation (1) is widely used. This equation is a key equation in the theory of micromagnetism [17].

Many authors have studied the regularity, uniqueness and existence of the solutions to the simplified version of (8) without the magnetic field $\boldsymbol{H}$. We mention papers [1,2,6–8,11,16] for the results in this area. For the overview of numerical methods used in micromagnetism we refer to a review paper [14].

The coupled Maxwell–Landau–Lifshitz (M-LL) system with space-periodic boundary conditions in two-dimensional case has been studied in [13], where the authors have proved the uniqueness of global smooth solution for the case of small initial energies. The extension of this work for finite initial energies can be found in [15]. We establish similar results for the three-dimensional case. However, our regularity and uniqueness results are not global. They are valid for finite time.

In 3D case, the existence of a weak solution was proved in [12].

Carbou and Fabrie studied the combination of the LL equation with Maxwell's equations in [5] considering the fields $\boldsymbol{E}$, $\boldsymbol{H}$ to be defined in the whole space $\mathbb{R}^3$ without any boundary conditions. They proved the existence of weak solutions to the full M-LL system, considering the exchange term in the LL equation. To do this they first introduced a relaxed problem with the penalization term $\lambda^{-1}(|\boldsymbol{m}|^2 - 1)$ which forces the length of the magnetization to remain constant as $\lambda \to 0$. This approach was already used by other authors, e.g., in [11]. It involves the construction of Galerkin approximations which in the limit converge to the solution of the original unrelaxed problem. Carbou and Fabrie did not discuss the uniqueness and regularity of the solutions.

The approach we use does not employ the penalization term. Our work is inspired by the works [13,15] where the authors consider the M-LL system with space-periodic solutions in lower dimensions.

The organization of the paper is as follows. In Section 3 we prove several lemmas in which we derive necessary a priori estimates for the smooth solution. Next, in Section 4 we non-trivially extend the existence result from [12]. Finally, in Section 5 we prove the uniqueness of the solution.

3. A priori estimates

In the following text, we use the symbol $(.,.)$ for the standard $L^2(\Omega)$ product. The norm in a general function space X is denoted by $\|.\|_X$. In particular, for the norm in $L^p(\Omega)$ we write $\|.\|_p$ for $\infty \geqslant p \geqslant 1$. For scalar product in $\mathbb{R}^3$ ($\mathbb{R}^m$, respectively) we use $\langle .,. \rangle$ ($\langle .,. \rangle_m$, respectively). In the entire text, the letters C and ε denote generic numbers, which may change if necessary.

Recall some integral inequalities derived from the extended Sobolev inequalities, for more details see [9]. For a bounded domain Ω with $\partial\Omega$ in C^2 we have

$$\|u\|_{L^4} \leqslant C\|u\|_{W^{1,2}}^{3/4}\|u\|_{L^2}^{1/4} \leqslant C\|u\|_{L^2} + C\|\nabla u\|_{L^2}^{3/4}\|u\|_{L^2}^{1/4}, \tag{9}$$

$$\|\nabla u\|_{L^4} \leqslant C\|\nabla u\|_{W^{1,2}}^{3/4}\|\nabla u\|_{L^2}^{1/4} \leqslant C\|\nabla u\|_{L^2} + C\|\Delta u\|_{L^2}^{3/4}\|\nabla u\|_{L^2}^{1/4}, \tag{10}$$

$$\|\Delta u\|_{L^4} \leqslant C\|\Delta u\|_{W^{1,2}}^{3/4}\|\Delta u\|_{L^2}^{1/4} \leqslant C\|\Delta u\|_{L^2} + C\|\nabla\Delta u\|_{L^2}^{3/4}\|\Delta u\|_{L^2}^{1/4}, \tag{11}$$

$$\|\nabla u\|_{W^{1,4}} \leqslant C\big[\|\nabla u\|_2 + \|\Delta u\|_2 + \big(\|\nabla u\|_2^{1/4} + \|\Delta u\|_2^{1/4}\big)\|\nabla\Delta u\|_2^{3/4}\big], \tag{12}$$

valid for any function u regular enough. The previous estimates can be of course used also in the case of vector valued function $\boldsymbol{u}$.

We recall also that under certain circumstances Eqs. (4), (5) can be derived directly from (2), (3), see Lemma 1. This comes from the identity $\nabla \cdot \nabla\times = 0$. Knowing this, we focus mostly on Eqs. (1)–(3). The proof of the following result for weak solutions can be found in [12].

Lemma 1. *Assume the initial value vector functions $\boldsymbol{E}_0(\boldsymbol{x}), \boldsymbol{H}_0(\boldsymbol{x}), \boldsymbol{m}_0(\boldsymbol{x})$ satisfy conditions*

$$(\nabla\vartheta_0, \boldsymbol{E}_0) = 0, \qquad (\nabla\vartheta_0, \boldsymbol{H}_0 + \beta\boldsymbol{m}_0) = 0,$$

where $\vartheta \in C^1((0,T)\times\Omega)$, $\vartheta(\boldsymbol{x},T) = 0$, $\vartheta_0 = \vartheta(\boldsymbol{x},0)$ are the functions appearing in the definition of a weak solution to Problem P, see [12,15]. *Then from* (2), (3) *it follows that*

$$\int_I (\boldsymbol{E}, \nabla\vartheta)\,\mathrm{d}s = 0, \qquad \int_I (\boldsymbol{H} + \beta\boldsymbol{m}, \nabla\vartheta)\,\mathrm{d}s = 0.$$

Now, in Lemmas 2–6 we derive a priori estimates for the exact solution of Problem P. Afterwards, using these estimates we will be able to settle appropriate regularity of the solutions $\boldsymbol{m}, \boldsymbol{E}, \boldsymbol{H}$ and their uniqueness.

Lemma 2. *Suppose that $\alpha > 0, \sigma \geqslant 0, \beta \geqslant 0$. Then for every real positive T the following estimates are valid for the solution $\boldsymbol{m}$ to Problem P*:

$$\partial_t \|\boldsymbol{m}\|_2^2 + \alpha\|\nabla\boldsymbol{m}\|_2^2 \leqslant C\big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18}\big).$$

Proof. Take (8), multiply by $\boldsymbol{m}$ and integrate over Ω to get

$$\frac{1}{2}\partial_t\|\boldsymbol{m}\|_2^2 + \alpha\|\nabla\boldsymbol{m}\|_2^2 \leqslant \alpha\big|\big(|\nabla\boldsymbol{m}|^2\boldsymbol{m}, \boldsymbol{m}\big)\big| \leqslant \alpha\|\nabla\boldsymbol{m}\|_2^2\|\boldsymbol{m}\|_{L^\infty}^2.$$

Next we use the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ to conclude

$$\partial_t\|\boldsymbol{m}\|_2^2 + \alpha\|\nabla\boldsymbol{m}\|_2^2 \leqslant C\|\boldsymbol{m}\|_{W^{2,2}}^4 \leqslant C\big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18}\big),$$

where we have used the Young inequality. We are able to prove better results with lower exponents than 18. However, we need such a high exponent in the following lemmas. □

Lemma 3. *Under the assumptions of Lemma* 2, *for every real positive T the following estimates are valid for the solution $\boldsymbol{m}, \boldsymbol{H}$ to Problem P*:

$$\partial_t\|\nabla\boldsymbol{m}\|_2^2 + \alpha\|\Delta\boldsymbol{m}\|_2^2 \leqslant C\big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18} + \|\boldsymbol{H}\|_2^4\big).$$

Proof. Take (8), multiply by $-\Delta\boldsymbol{m}$ and integrate over Ω to get

$$\frac{1}{2}\partial_t\|\nabla\boldsymbol{m}\|_2^2 + \alpha\|\Delta\boldsymbol{m}\|_2^2 \leqslant \alpha\|\nabla\boldsymbol{m}\|_4\|\nabla\boldsymbol{m}\|_4\|\boldsymbol{m}\|_{L^\infty}\|\Delta\boldsymbol{m}\|_2 + \big|(\boldsymbol{m}\times\Delta\boldsymbol{m}, \Delta\boldsymbol{m})\big|$$
$$+ \|\boldsymbol{m}\|_{L^\infty}\|\boldsymbol{H}\|_2\|\Delta\boldsymbol{m}\|_2 + \|\boldsymbol{m}\|_{L^\infty}^2\|\boldsymbol{H}\|_2\|\Delta\boldsymbol{m}\|_2.$$

We use the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Young's inequality to conclude

$$\partial_t\|\nabla\boldsymbol{m}\|_2^2 + \alpha\|\Delta\boldsymbol{m}\|_2^2 \leqslant C\big(\|\boldsymbol{m}\|_{W^{2,2}}^6 + \|\boldsymbol{m}\|_{W^{2,2}}^4 + \|\boldsymbol{H}\|_2^2\big),$$

which verifies the result of the lemma. □

Lemma 4. *Under the assumptions of Lemma* 2, *for every real positive T the following estimates are valid for the solution $\boldsymbol{m}$, $\boldsymbol{H}$ to Problem P*:

$$\partial_t\|\Delta\boldsymbol{m}\|_2^2 + \alpha\|\nabla\Delta\boldsymbol{m}\|_2^2 \leqslant C\big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18} + \|\boldsymbol{H}\|_{W^{1,2}}^4\big).$$

Proof. We multiply (8) by $\Delta^2\boldsymbol{m}$, integrate over Ω and afterwards perform integration by parts. Thanks to the periodicity of the functions we get rid of all boundary terms and we arrive at

$$\begin{aligned}
&(\partial_t\Delta\boldsymbol{m}, \Delta\boldsymbol{m}) + \alpha\|\nabla\Delta\boldsymbol{m}\|_2^2 \\
&\quad\leqslant \alpha\big|\big(\nabla\big(|\nabla\boldsymbol{m}|^2\boldsymbol{m}\big), \nabla\Delta\boldsymbol{m}\big)\big| + \big|\big(\nabla(\boldsymbol{m}\times\Delta\boldsymbol{m}), \nabla\Delta\boldsymbol{m}\big)\big| \\
&\qquad + \big|\big(\nabla(\boldsymbol{m}\times\boldsymbol{H}), \nabla\Delta\boldsymbol{m}\big)\big| + \alpha\big|\big(\nabla\big(\boldsymbol{m}\times(\boldsymbol{m}\times\boldsymbol{H})\big), \nabla\Delta\boldsymbol{m}\big)\big|.
\end{aligned}$$

Next we derive the terms on the right-hand side. Then we use appropriate integral inequalities to obtain

$$\begin{aligned}
&\partial_t\|\Delta\boldsymbol{m}\|_2^2 + \alpha\|\nabla\Delta\boldsymbol{m}\|_2^2 \\
&\quad\leqslant 2\|\nabla\boldsymbol{m}\|_{W^{1,4}}\|\nabla\boldsymbol{m}\|_4\|\boldsymbol{m}\|_{L^\infty}\|\nabla\Delta\boldsymbol{m}\|_2 + \|\nabla\boldsymbol{m}\|_4^2\|\nabla\boldsymbol{m}\|_{L^\infty}\|\nabla\Delta\boldsymbol{m}\|_2 \\
&\qquad + \|\nabla\boldsymbol{m}\|_4\|\Delta\boldsymbol{m}\|_4\|\nabla\Delta\boldsymbol{m}\|_2 + \big|(\boldsymbol{m}\times\nabla\Delta\boldsymbol{m}, \nabla\Delta\boldsymbol{m})\big| \\
&\qquad + \|\nabla\boldsymbol{m}\|_4\|\boldsymbol{H}\|_4\|\nabla\Delta\boldsymbol{m}\|_2 + \|\boldsymbol{m}\|_{L^\infty}\|\nabla\boldsymbol{H}\|_2\|\nabla\Delta\boldsymbol{m}\|_2 \\
&\qquad + 2\|\nabla\boldsymbol{m}\|_4\|\boldsymbol{m}\|_{L^\infty}\|\boldsymbol{H}\|_4\|\nabla\Delta\boldsymbol{m}\|_2 + \|\boldsymbol{m}\|_{L^\infty}^2\|\nabla\boldsymbol{H}\|_2\|\nabla\Delta\boldsymbol{m}\|_2.
\end{aligned}$$

Notice that the term $(\boldsymbol{m}\times\nabla\Delta\boldsymbol{m}, \nabla\Delta\boldsymbol{m}) = 0$. To proceed we use embeddings $W^{2,2}(\Omega)\hookrightarrow L^\infty(\Omega)$ and $W^{1,4}(\Omega)\hookrightarrow L^\infty(\Omega)$ to arrive at

$$\begin{aligned}
&\partial_t\|\Delta\boldsymbol{m}\|_2^2 + \alpha\|\nabla\Delta\boldsymbol{m}\|_2^2 \\
&\quad\leqslant 2\|\nabla\boldsymbol{m}\|_{W^{1,4}}\|\nabla\boldsymbol{m}\|_4\|\boldsymbol{m}\|_{W^{2,2}}\|\nabla\Delta\boldsymbol{m}\|_2 + \|\nabla\boldsymbol{m}\|_4^2\|\nabla\boldsymbol{m}\|_{W^{1,4}}\|\nabla\Delta\boldsymbol{m}\|_2 \\
&\qquad + \|\nabla\boldsymbol{m}\|_4\|\Delta\boldsymbol{m}\|_4\|\nabla\Delta\boldsymbol{m}\|_2 + \|\nabla\boldsymbol{m}\|_4\|\boldsymbol{H}\|_4\|\nabla\Delta\boldsymbol{m}\|_2 \\
&\qquad + \|\boldsymbol{m}\|_{W^{2,2}}\|\nabla\boldsymbol{H}\|_2\|\nabla\Delta\boldsymbol{m}\|_2 + 2\|\nabla\boldsymbol{m}\|_4\|\boldsymbol{m}\|_{W^{2,2}}\|\boldsymbol{H}\|_4\|\nabla\Delta\boldsymbol{m}\|_2 \\
&\qquad + \|\boldsymbol{m}\|_{W^{2,2}}^2\|\nabla\boldsymbol{H}\|_2\|\nabla\Delta\boldsymbol{m}\|_2.
\end{aligned}$$

We get rid of the term $\|\nabla\Delta\boldsymbol{m}\|_2$ using weighted Young inequality

$$\begin{aligned}
&\partial_t\|\Delta\boldsymbol{m}\|_2^2 + \alpha\|\nabla\Delta\boldsymbol{m}\|_2^2 \\
&\quad\leqslant \varepsilon\|\nabla\Delta\boldsymbol{m}\|_2^2 + C_\varepsilon\|\nabla\boldsymbol{m}\|_{W^{1,4}}^2\|\nabla\boldsymbol{m}\|_4^2\|\boldsymbol{m}\|_{W^{2,2}}^2 + C_\varepsilon\|\nabla\boldsymbol{m}\|_4^4\|\nabla\boldsymbol{m}\|_{W^{1,4}}^2 \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{m}\|_4^2\|\Delta\boldsymbol{m}\|_4^2 + C_\varepsilon\|\nabla\boldsymbol{m}\|_4^2\|\boldsymbol{H}\|_4^2 + C_\varepsilon\|\nabla\boldsymbol{H}\|_2^2\|\boldsymbol{m}\|_{W^{2,2}}^2 \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{m}\|_4^2\|\boldsymbol{H}\|_4^2\|\boldsymbol{m}\|_{W^{2,2}}^2 + C_\varepsilon\|\nabla\boldsymbol{H}\|_2^2\|\boldsymbol{m}\|_{W^{2,2}}^4.
\end{aligned}$$

We make use of (12), (10) and (11) to continue

$$\begin{aligned}
&\partial_t\|\Delta\boldsymbol{m}\|_2^2 + \alpha\|\nabla\Delta\boldsymbol{m}\|_2^2 \\
&\quad\leqslant \varepsilon\|\nabla\Delta\boldsymbol{m}\|_2^2 + C_\varepsilon\|\Delta\boldsymbol{m}\|_2^{1/2}\|\nabla\Delta\boldsymbol{m}\|_2^{3/2}\|\nabla\boldsymbol{m}\|_2^{1/2}\|\Delta\boldsymbol{m}\|_2^{3/2}\|\boldsymbol{m}\|_{W^{2,2}}^2 \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{m}\|_2\|\Delta\boldsymbol{m}\|_2^3\|\Delta\boldsymbol{m}\|_2^{1/2}\|\nabla\Delta\boldsymbol{m}\|_2^{3/2} \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{m}\|_2^{1/2}\|\Delta\boldsymbol{m}\|_2^{3/2}\|\Delta\boldsymbol{m}\|_2^{1/2}\|\nabla\Delta\boldsymbol{m}\|_2^{3/2} \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{m}\|_2^{1/2}\|\Delta\boldsymbol{m}\|_2^{3/2}\|\boldsymbol{H}\|_2^{1/2}\|\nabla\boldsymbol{H}\|_2^{3/2} + C_\varepsilon\|\nabla\boldsymbol{H}\|_2^4 + \|\boldsymbol{m}\|_{W^{2,2}}^4 \\
&\qquad + C_\varepsilon\|\nabla\boldsymbol{m}\|_2^{1/2}\|\Delta\boldsymbol{m}\|_2^{3/2}\|\boldsymbol{H}\|_2^{1/2}\|\nabla\boldsymbol{H}\|_2^{3/2}\|\boldsymbol{m}\|_{W^{2,2}}^2 + C_\varepsilon\|\nabla\boldsymbol{H}\|_2^4 \\
&\qquad + \|\boldsymbol{m}\|_{W^{2,2}}^8.
\end{aligned}$$

In the previous expression we skipped some terms. We considered only terms including the highest space derivative as the “worst” case. The terms with lower space derivative would not cause any problems.

After using weighted Young inequality and setting ε small enough we get

$$\partial_t \|\Delta \boldsymbol{m}\|_2^2 + \|\nabla \Delta \boldsymbol{m}\|_2^2 \leqslant C_\varepsilon \big(1 + \|\boldsymbol{H}\|_2^4 + \|\nabla \boldsymbol{H}\|_2^4 + \|\boldsymbol{m}\|_{W^{2,2}}^{18}\big),$$

which verifies the result of the lemma. □

Lemma 5. *Under the assumptions of Lemma* 2*, for every real positive T the following estimates are valid for the solution $\boldsymbol{m}$, $\boldsymbol{H}$, $\boldsymbol{E}$ to Problem P*:

$$\partial_t \|\boldsymbol{E}\|_2^2 + \partial_t \|\boldsymbol{H}\|_2^2 \leqslant C\big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18} + \|\boldsymbol{H}\|_2^4\big).$$

Proof. Take (2) and (3), multiply by $\boldsymbol{E}$, $\boldsymbol{H}$, respectively, and integrate over Ω to get

$$(\partial_t \boldsymbol{E}, \boldsymbol{E}) + \sigma \|\boldsymbol{E}\|_2^2 - (\nabla \times \boldsymbol{H}, \boldsymbol{E}) = 0, \tag{13}$$

$$(\partial_t \boldsymbol{H}, \boldsymbol{H}) + (\nabla \times \boldsymbol{E}, \boldsymbol{H}) = -(\beta \partial_t \boldsymbol{m}, \boldsymbol{H}). \tag{14}$$

Because of the boundary conditions (6) we have

$$(\nabla \times \boldsymbol{H}, \boldsymbol{E}) - (\nabla \times \boldsymbol{E}, \boldsymbol{H}) = 0.$$

Then after summing up Eqs. (13)–(14) we get

$$\partial_t \|\boldsymbol{E}\|_2^2 + \partial_t \|\boldsymbol{H}\|_2^2 \leqslant \beta \big|(\partial_t \boldsymbol{m}, \boldsymbol{H})\big|. \tag{15}$$

Using (8) we estimate the term $|(\partial_t \boldsymbol{m}, \boldsymbol{H})|$ as

$$\begin{aligned}\big|(\partial_t \boldsymbol{m}, \boldsymbol{H})\big| \leqslant{}& \alpha \big|(\Delta \boldsymbol{m}, \boldsymbol{H})\big| + \alpha \big|\big(|\nabla \boldsymbol{m}|^2 \boldsymbol{m}, \boldsymbol{H}\big)\big| \\ &+ \big|(\boldsymbol{m} \times \Delta \boldsymbol{m}, \boldsymbol{H})\big| + \big|(\boldsymbol{m} \times \boldsymbol{H}, \boldsymbol{H})\big| \\ &+ \alpha \big|\big(\boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{H}), \boldsymbol{H}\big)\big|.\end{aligned}$$

We use appropriate integral inequalities to get

$$\begin{aligned}\big|(\partial_t \boldsymbol{m}, \boldsymbol{H})\big| \leqslant{}& \alpha \|\Delta \boldsymbol{m}\|_2 \|\boldsymbol{H}\|_2 + \alpha \|\nabla \boldsymbol{m}\|_4^2 \|\boldsymbol{m}\|_{L^\infty} \|\boldsymbol{H}\|_2 \\ &+ \|\boldsymbol{m}\|_{L^\infty} \|\Delta \boldsymbol{m}\|_2 \|\boldsymbol{H}\|_2 + \alpha \|\boldsymbol{m}\|_{L^\infty}^2 \|\boldsymbol{H}\|_2^2.\end{aligned}$$

We use the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ and Young's inequality to proceed and get

$$\big|(\partial_t \boldsymbol{m}, \boldsymbol{H})\big| \leqslant C\big[\|\boldsymbol{m}\|_{W^{2,2}}^2 + \|\boldsymbol{H}\|_2^2 + \|\boldsymbol{m}\|_{W^{2,2}}^6 + \|\boldsymbol{m}\|_{W^{2,2}}^4 + \|\boldsymbol{H}\|_2^4\big].$$

Then we again apply the Young inequality to verify that

$$\big|(\partial_t \boldsymbol{m}, \boldsymbol{H})\big| \leqslant C\big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18} + \|\boldsymbol{H}\|_2^4\big).$$

The previous result together with (15) concludes the proof. □

Lemma 6. *Under the assumptions of Lemma* 2*, for every real positive T the following estimates are valid for the solution $\boldsymbol{m}$, $\boldsymbol{H}$, $\boldsymbol{E}$ to Problem P*:

$$\partial_t \|\nabla \boldsymbol{E}\|_2^2 + \partial_t \|\nabla \boldsymbol{H}\|_2^2 \leqslant C_\varepsilon \big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18} + \|\boldsymbol{H}\|_{W^{1,2}}^4 + \|\boldsymbol{E}\|_{W^{1,2}}^2\big) + \varepsilon \|\nabla \Delta \boldsymbol{m}\|_2^2.$$

Proof. We multiply (2) and (3) with $-\Delta \boldsymbol{E}$ and $-\Delta \boldsymbol{H}$, respectively. Considering the periodicity of the functions we arrive at

$$\partial_t \|\nabla \boldsymbol{E}\|_2^2 + \partial_t \|\nabla \boldsymbol{H}\|_2^2 \leqslant \left|(\partial_t \nabla \boldsymbol{m}, \nabla \boldsymbol{H})\right| + \left|(\nabla \times \boldsymbol{E}, -\Delta \boldsymbol{H}) - (\nabla \times \boldsymbol{H}, -\Delta \boldsymbol{E})\right|. \tag{16}$$

First, using the identity $\nabla \times \nabla \times \boldsymbol{u} = -\Delta \boldsymbol{u} + \nabla\nabla \cdot \boldsymbol{u}$ we compute

$$\begin{aligned}(\nabla \times \boldsymbol{H}, -\Delta \boldsymbol{E}) &= (\nabla \times \boldsymbol{H}, \nabla \times \nabla \times \boldsymbol{E}) - (\nabla \times \boldsymbol{H}, \nabla\nabla \cdot \boldsymbol{E}) \\ &= (\nabla \times \boldsymbol{H}, \nabla \times \nabla \times \boldsymbol{E}),\end{aligned}$$

since $\nabla \cdot \boldsymbol{E} = 0$ from (5). We employed the periodicity of the functions. Analogously we get

$$\begin{aligned}(\nabla \times \boldsymbol{E}, -\Delta \boldsymbol{H}) &= (\nabla \times \boldsymbol{E}, \nabla \times \nabla \times \boldsymbol{H}) - \big(\nabla \times \boldsymbol{E}, \nabla\nabla \cdot (\boldsymbol{H} + \beta \boldsymbol{m})\big) \\ &\quad + (\nabla \times \boldsymbol{E}, \beta \nabla\nabla \cdot \boldsymbol{m}) \\ &= (\nabla \times \boldsymbol{E}, \nabla \times \nabla \times \boldsymbol{H}) + (\nabla \times \boldsymbol{E}, \beta \nabla\nabla \cdot \boldsymbol{m}),\end{aligned}$$

since $\nabla \cdot (\boldsymbol{H} + \beta \boldsymbol{m}) = 0$ from (5). Adding the previous two identities to (16) we estimate

$$\partial_t \|\nabla \boldsymbol{E}\|_2^2 + \partial_t \|\nabla \boldsymbol{H}\|_2^2 \leqslant \left|(\partial_t \nabla \boldsymbol{m}, \nabla \boldsymbol{H})\right| + \|\nabla \boldsymbol{E}\|_2^2 + \|\boldsymbol{m}\|_{W^{2,2}}^2.$$

We multiply (8) with $-\Delta \boldsymbol{H}$ in order to estimate the term $|(\partial_t \nabla \boldsymbol{m}, \nabla \boldsymbol{H})|$. We get

$$\begin{aligned}&\left|(\partial_t \nabla \boldsymbol{m}, \nabla \boldsymbol{H})\right| \\ &\quad \leqslant \alpha \left|(\nabla \Delta \boldsymbol{m}, \nabla \boldsymbol{H})\right| + \alpha \left|\big(\nabla\big(|\nabla \boldsymbol{m}|^2 \boldsymbol{m}\big), \nabla \boldsymbol{H}\big)\right| + \left|\big(\nabla(\boldsymbol{m} \times \boldsymbol{H}), \nabla \boldsymbol{H}\big)\right| \\ &\qquad + \left|\big(\nabla(\boldsymbol{m} \times \Delta \boldsymbol{m}), \nabla \boldsymbol{H}\big)\right| + \alpha \left|\big(\nabla\big(\boldsymbol{m} \times (\boldsymbol{m} \times \boldsymbol{H})\big), \nabla \boldsymbol{H}\big)\right| \\ &\quad \leqslant \varepsilon \|\nabla \Delta \boldsymbol{m}\|_2^2 + C_\varepsilon \|\nabla \boldsymbol{H}\|_2^2 + 2\|\nabla \boldsymbol{m}\|_{W^{1,4}} \|\nabla \boldsymbol{m}\|_4 \|\boldsymbol{m}\|_{L^\infty} \|\nabla \boldsymbol{H}\|_2 \\ &\qquad + \|\nabla \boldsymbol{m}\|_4^2 \|\nabla \boldsymbol{m}\|_{L^\infty} \|\nabla \boldsymbol{H}\|_2 + \|\nabla \boldsymbol{m}\|_4 \|\Delta \boldsymbol{m}\|_4 \|\nabla \boldsymbol{H}\|_2 \\ &\qquad + \|\boldsymbol{m}\|_{L^\infty} \|\nabla \Delta \boldsymbol{m}\|_2 \|\nabla \boldsymbol{H}\|_2 + \|\nabla \boldsymbol{m}\|_4 \|\boldsymbol{H}\|_4 \|\nabla \boldsymbol{H}\|_2 \\ &\qquad + \|\boldsymbol{m}\|_{L^\infty} \|\nabla \boldsymbol{H}\|_2^2 + 2\|\nabla \boldsymbol{m}\|_4 \|\boldsymbol{m}\|_{L^\infty} \|\boldsymbol{H}\|_4 \|\nabla \boldsymbol{H}\|_2 \\ &\qquad + \|\boldsymbol{m}\|_{L^\infty}^2 \|\nabla \boldsymbol{H}\|_2^2.\end{aligned}$$

Next we use (12), the embeddings $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, and the Young inequality for several times to arrive at

$$\begin{aligned}\left|(\partial_t \nabla \boldsymbol{m}, \nabla \boldsymbol{H})\right| \leqslant \varepsilon \|\nabla \Delta \boldsymbol{m}\|_2^2 + C_\varepsilon \big[&\|\nabla \boldsymbol{H}\|_2^2 + \|\Delta \boldsymbol{m}\|_4^2 + \|\boldsymbol{m}\|_{W^{2,2}}^4 \|\nabla \boldsymbol{H}\|_2^2 \\ &+ \|\boldsymbol{m}\|_{W^{2,2}}^2 \|\nabla \boldsymbol{H}\|_2^2 + \|\boldsymbol{m}\|_{W^{2,2}}^2 + \|\boldsymbol{H}\|_{W^{1,2}}^4 + \|\nabla \boldsymbol{H}\|_2^4 + \|\boldsymbol{m}\|_{W^{2,2}}^4\big].\end{aligned}$$

Using (11) and again the Young inequality gives us

$$\left|(\partial_t \nabla \boldsymbol{m}, \nabla \boldsymbol{H})\right| \leqslant \varepsilon \|\nabla \Delta \boldsymbol{m}\|_2^2 + C_\varepsilon \big[1 + \|\boldsymbol{m}\|_{W^{2,2}}^8 + \|\boldsymbol{H}\|_{W^{1,2}}^4 + \|\boldsymbol{E}\|_{W^{1,2}}^2\big],$$

which confirms the result of the lemma. □

If we summarize the results from Lemmas 2–6, we get

$$\begin{aligned}&\partial_t \big(\|\boldsymbol{m}\|_2^2 + \|\Delta \boldsymbol{m}\|_2^2\big) + \partial_t \big(\|\boldsymbol{E}\|_{W^{1,2}}^2 + \|\boldsymbol{H}\|_{W^{1,2}}^2\big) + \alpha \|\nabla \Delta \boldsymbol{m}\|_2^2 \\ &\quad \leqslant C_\varepsilon \big(1 + \|\boldsymbol{m}\|_{W^{2,2}}^{18} + \|\boldsymbol{E}\|_{W^{1,2}}^4 + \|\boldsymbol{H}\|_{W^{1,2}}^4\big) + \varepsilon \|\nabla \Delta \boldsymbol{m}\|_2^2.\end{aligned} \tag{17}$$

As ε was arbitrary positive real number, suppose $\varepsilon = \alpha/2$. Then, we easily absorb the term $\varepsilon \|\nabla \Delta \boldsymbol{m}\|_2^2$ on the left-hand side.

Considering Ω as a domain without boundary we have the equivalence of the norms $\|u\|_{W^{2,2}}$ and $(\|u\|_2^2 + \|\Delta u\|_2^2)^{1/2}$. Then, we use this result in (17) to replace $\|\boldsymbol{m}\|_{W^{2,2}}$ with $\|\boldsymbol{m}\|_2^2 + \|\Delta \boldsymbol{m}\|_2^2$. We get

$$\begin{aligned}
&\partial_t\big(\|\boldsymbol{m}\|_2^2 + \|\Delta \boldsymbol{m}\|_2^2\big) + \frac{\alpha}{2}\|\nabla\Delta \boldsymbol{m}\|_2^2 + \partial_t\|\boldsymbol{E}\|_{W^{1,2}}^2 + \partial_t\|\boldsymbol{H}\|_{W^{1,2}}^2 \\
&\quad \leqslant C\big(1 + \|\boldsymbol{m}\|_2^{18} + \|\Delta \boldsymbol{m}\|_2^{18} + \|\boldsymbol{E}\|_{W^{1,2}}^4 + \|\boldsymbol{H}\|_{W^{1,2}}^4\big).
\end{aligned} \tag{18}$$

4. Regularity of the solution

Let us define the following functions:

$$f(t) = \|\boldsymbol{m}\|_2^2 + \|\Delta \boldsymbol{m}\|_2^2,$$
$$g(t) = \|\boldsymbol{E}\|_{W^{1,2}}^2 + \|\boldsymbol{H}\|_{W^{1,2}}^2.$$

From (18) we can then conclude that

$$f'(t) + g'(t) \leqslant C\big(1 + f^9(t) + g^2(t)\big).$$

In the next lemma we recall a Bihary-type inequality, see, for example, [3].

Lemma 7. *Let u, a, b and k be non-negative continuous functions in $J = [\alpha_1, \beta_1]$, and let $p > 1$ be a constant. Suppose a/b is non-decreasing in J and*

$$u(t) \leqslant a(t) + b(t)\int_{\alpha_1}^{\beta_1} k(s)u^p(s)\,\mathrm{d}s, \quad t \in J.$$

Then

$$u(t) \leqslant a(t)\left\{1 - (p-1)\int_{\alpha_1}^{\beta_1} k(s)b(s)a^{p-1}(s)\,\mathrm{d}s\right\}^{\frac{1}{1-p}}, \quad \alpha \leqslant t \leqslant \beta_p,$$

where $\beta_p = \sup\{t \in J:\ (p-1)\int_{\alpha_1}^{t} k(s)b(s)a^{p-1}(s)\,\mathrm{d}s < 1\}$.

Using this lemma, in the following theorem we state the in-time local regularity result for the solution $(\boldsymbol{m}, \boldsymbol{E}, \boldsymbol{H})$ to the system (8), (2)–(5), (6), (7).

Theorem 1. *Assume space-periodic initial value functions $\boldsymbol{m}_0 \in \boldsymbol{W}_{\mathrm{per}}^{2,2}(\Omega)$, $\boldsymbol{H}_0, \boldsymbol{E}_0 \in \boldsymbol{W}_{\mathrm{per}}^{1,2}(\Omega)$. Then for smooth solution $\boldsymbol{m}, \boldsymbol{E}, \boldsymbol{H}$ of the Problem P there exist a positive T_0 and a constant C such that for every positive $T < T_0$ it is valid*

$$\sup_{0<t<T}\big[\|\boldsymbol{m}\|_{W^{2,2}}^2 + \|\boldsymbol{E}\|_{W^{1,2}}^2 + \|\boldsymbol{H}\|_{W^{1,2}}^2\big] \leqslant C, \tag{19}$$

$$\sup_{0<t<T}\|\boldsymbol{m}\|_{L^\infty}^2 \leqslant C, \tag{20}$$

$$\int_0^T \|\boldsymbol{m}\|_{W^{3,2}}^2\,\mathrm{d}\tau \leqslant C. \tag{21}$$

Numbers T_0 and C both depend only on the domain Ω, parameter α, on the size of initial data $\boldsymbol{m}_0$ in $\boldsymbol{W}_{\mathrm{per}}^{2,2}(\Omega)$, and on the size of initial data $\boldsymbol{H}_0, \boldsymbol{E}_0$ in $\boldsymbol{W}_{\mathrm{per}}^{1,2}(\Omega)$.

Proof. The first result comes directly using Lemma 7 for functions f and g. The second result is a simple consequence of the first one and the embedding $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$. The third result can be verified by integrating (18) in time and using (19). □

Lemma 8. *Under the assumptions of Theorem* 1 *we get the estimate*

$$\int_0^T \|\partial_t \boldsymbol{m}\|^2_{W^{1,2}} \leqslant C.$$

Proof. Multiply (8) first with $\partial_t \boldsymbol{m}$ and then with $-\partial_t \Delta \boldsymbol{m}$ to verify the result of the lemma. □

Existence

In the previous text we have been talking in terms of "for the smooth solution ... it is valid ...." Further we will prove the *existence* of the solutions to Problem P. By proving the existence of the solutions with higher regularity we extend the following results from [12].

Theorem 2. (3.1 from [12]) *Assume* $\boldsymbol{m}_0 \in W^{1,2}(\Omega)$, $\boldsymbol{H}_0, \boldsymbol{E}_0 \in L^2(\Omega)$, *and they are periodic functions with periodicity D, satisfying the assumptions from Lemma* 1. *Then the periodic initial value Problem P has at least one weak solution* $\boldsymbol{m}, \boldsymbol{H}, \boldsymbol{E}$ *satisfying*

$$\begin{aligned}
&\boldsymbol{m}(x,t) \in L^\infty\big(0,T; W^{1,2}(\Omega)\big),\\
&\boldsymbol{H}(x,t), \boldsymbol{E}(x,t) \in L^\infty\big(0,T; L^2(\Omega)\big).
\end{aligned}$$

The previous existence result was established using the Galerkin method. The essential estimate for the finite-dimensional approximate solution appearing in the Galerkin method was

$$\sup_{t\in I}\big[\|\boldsymbol{m}_N(.,t)\|^2_{W^{1,2}} + \|\boldsymbol{H}_N(.,t)\|^2_{L^2} + \|\boldsymbol{E}_N(.,t)\|^2_{L^2}\big] \leqslant C. \tag{22}$$

Notice that governing equations for this approximate solution $\boldsymbol{m}_N, \boldsymbol{H}_N, \boldsymbol{E}_N$ are identical with the equations from Problem P. Therefore, using the results from Theorem 1 we are able to get the following higher regularity result for the approximate solution:

$$\sup_{t\in I}\big[\|\boldsymbol{m}_N(.,t)\|^2_{W^{2,2}} + \|\boldsymbol{H}_N(.,t)\|^2_{W^{1,2}} + \|\boldsymbol{E}_N(.,t)\|^2_{W^{1,2}}\big] \leqslant C, \tag{23}$$

assuming higher regularity of the initial conditions. Consequently we can state the following existence theorem:

Theorem 3. *Assume* $\boldsymbol{m}_0 \in W^{2,2}(\Omega)$, $\boldsymbol{H}_0, \boldsymbol{E}_0 \in W^{1,2}(\Omega)$, *and they are periodic functions with periodicity D, and satisfy assumptions from Lemma* 1. *Then there exists a positive* T_0 *such that the periodic initial value Problem P has at least one weak solution* $\boldsymbol{m}, \boldsymbol{H}, \boldsymbol{E}$ *satisfying*

$$\begin{aligned}
&\boldsymbol{m}(x,t) \in L^\infty\big(0,T_0; W^{2,2}(\Omega)\big),\\
&\boldsymbol{H}(x,t), \boldsymbol{E}(x,t) \in L^\infty\big(0,T_0; W^{1,2}(\Omega)\big).
\end{aligned}$$

Proof. The proof uses the same argumentation as the proof of Theorem 2, therefore we skip it. □

5. Uniqueness

The aim of this section is to prove the uniqueness of the solution on the interval $\langle 0, T_0\rangle$, where T_0 is from Theorem 1. In Theorem 4 we state the continuous dependence of the solution on the initial data. Uniqueness is then only a simple consequence of this theorem.

Theorem 4. *Suppose that* $(\boldsymbol{m}_1, \boldsymbol{E}_1, \boldsymbol{H}_1)$ *and* $(\boldsymbol{m}_2, \boldsymbol{E}_2, \boldsymbol{H}_2)$ *are two solutions of the system* (8)*,* (2)–(5)*. Denote*

$$\bar{\boldsymbol{m}} = \boldsymbol{m}_1 - \boldsymbol{m}_2, \qquad \bar{\boldsymbol{E}} = \boldsymbol{E}_1 - \boldsymbol{E}_2, \qquad \bar{\boldsymbol{H}} = \boldsymbol{H}_1 - \boldsymbol{H}_2.$$

Then the following dependence on the initial conditions holds:

$$\sup_{0<t<T_0} \big[\|\bar{\boldsymbol{m}}(t)\|_{W^{1,2}} + \|\bar{\boldsymbol{E}}(t)\|_2 + \|\bar{\boldsymbol{H}}(t)\|_2\big] \leqslant C\big[\|\bar{\boldsymbol{m}}(0)\|_{W^{1,2}} + \|\bar{\boldsymbol{E}}(0)\|_2 + \|\bar{\boldsymbol{H}}(0)\|_2\big].$$

Proof. The solutions $\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{H}_1, \boldsymbol{H}_2$ satisfy (8). The differences $\bar{\boldsymbol{m}}, \bar{\boldsymbol{H}}$ then satisfy

$$\begin{aligned} \partial_t \bar{\boldsymbol{m}} - \alpha \Delta \bar{\boldsymbol{m}} = {} & \alpha |\nabla \boldsymbol{m}_1|^2 \bar{\boldsymbol{m}} + \alpha\big(|\nabla \boldsymbol{m}_1|^2 - |\nabla \boldsymbol{m}_2|^2\big)\boldsymbol{m}_2 - \bar{\boldsymbol{m}} \times \Delta \boldsymbol{m}_1 \\ & - \boldsymbol{m}_2 \times \Delta \bar{\boldsymbol{m}} - \bar{\boldsymbol{m}} \times \boldsymbol{H}_1 - \boldsymbol{m}_2 \times \bar{\boldsymbol{H}} \\ & - \alpha\big[\bar{\boldsymbol{m}} \times (\boldsymbol{m}_1 \times \boldsymbol{H}_1) + \boldsymbol{m}_2 \times (\bar{\boldsymbol{m}} \times \boldsymbol{H}_1) \\ & + \boldsymbol{m}_2 \times (\boldsymbol{m}_2 \times \bar{\boldsymbol{H}})\big]. \end{aligned} \tag{24}$$

Taking the previous equation, multiplying by $\bar{\boldsymbol{m}}$ and integrating over Ω we directly get rid of the terms $\bar{\boldsymbol{m}} \times \boldsymbol{H}_1$, $\bar{\boldsymbol{m}} \times \Delta \boldsymbol{m}_1$ and $\bar{\boldsymbol{m}} \times (\boldsymbol{m}_1 \times \boldsymbol{H}_1)$. Remaining equation reads as

$$\begin{aligned} \frac{1}{2}\partial_t \|\bar{\boldsymbol{m}}\|_2^2 + \alpha \|\nabla \bar{\boldsymbol{m}}\|_2^2 = {} & \alpha \int_\Omega |\nabla \boldsymbol{m}_1|^2 \langle \bar{\boldsymbol{m}}, \bar{\boldsymbol{m}}\rangle + \int_\Omega \langle \nabla \bar{\boldsymbol{m}}, \nabla \boldsymbol{m}_1 + \nabla \boldsymbol{m}_2\rangle \langle \boldsymbol{m}_2, \bar{\boldsymbol{m}}\rangle \\ & - \int_\Omega \langle \boldsymbol{m}_2 \times \Delta \bar{\boldsymbol{m}}, \bar{\boldsymbol{m}}\rangle - \int_\Omega \langle \boldsymbol{m}_2 \times \bar{\boldsymbol{H}}, \bar{\boldsymbol{m}}\rangle + \int_\Omega \big\langle \boldsymbol{m}_2 \times (\bar{\boldsymbol{m}} \times \boldsymbol{H}_1), \bar{\boldsymbol{m}}\big\rangle \\ & + \int_\Omega \big\langle \boldsymbol{m}_2 \times (\boldsymbol{m}_2 \times \bar{\boldsymbol{H}}, \bar{\boldsymbol{m}})\big\rangle. \end{aligned}$$

Next, we perform integration by parts in the term including $\boldsymbol{m}_2 \times \Delta \bar{\boldsymbol{m}}$ and then we use integral inequalities to obtain

$$\begin{aligned} & \frac{1}{2}\partial_t \|\bar{\boldsymbol{m}}\|_2^2 + \alpha \|\nabla \bar{\boldsymbol{m}}\|_2^2 \\ & \quad \leqslant \alpha \|\nabla \boldsymbol{m}_1\|_4^2 \|\bar{\boldsymbol{m}}\|_4^2 + \|\nabla \bar{\boldsymbol{m}}\|_2 \big(\|\nabla \boldsymbol{m}_1\|_4 + \|\nabla \boldsymbol{m}_2\|_4\big) \|\boldsymbol{m}_2\|_{L^\infty} \|\bar{\boldsymbol{m}}\|_4 \\ & \qquad + \|\nabla \boldsymbol{m}_2\|_4 \|\nabla \bar{\boldsymbol{m}}\|_2 \|\bar{\boldsymbol{m}}\|_4 + \|\boldsymbol{m}_2\|_4 \|\bar{\boldsymbol{H}}\|_2 \|\bar{\boldsymbol{m}}\|_4 + \|\boldsymbol{m}_2\|_{L^\infty} \|\bar{\boldsymbol{m}}\|_4^2 \|\boldsymbol{H}_1\|_2 \\ & \qquad + \|\boldsymbol{m}_2\|_{L^\infty}^2 \|\bar{\boldsymbol{H}}\|_2 \|\bar{\boldsymbol{m}}\|_2. \end{aligned}$$

From the embeddings $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ we see for $i = 1, 2$ that

$$\begin{aligned} & \|\nabla \boldsymbol{m}_i\|_4 \leqslant C\|\boldsymbol{m}_i\|_{W^{2,2}}, \\ & \|\boldsymbol{m}_i\|_{L^\infty} \leqslant C\|\boldsymbol{m}_i\|_{W^{2,2}}. \end{aligned}$$

Further, using (19), (9), (10) and the Young inequality we arrive at

$$\partial_t \|\bar{\boldsymbol{m}}\|_2^2 + \|\nabla\bar{\boldsymbol{m}}\|_2^2 \leqslant \varepsilon\|\nabla\bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon\|\bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon\|\bar{\boldsymbol{H}}\|_2^2.$$

Setting ε small enough and integrating the equation in time we get

$$\left\|\bar{\boldsymbol{m}}(t)\right\|_2^2 - \left\|\bar{\boldsymbol{m}}(0)\right\|_2^2 + \int_0^t \left\|\nabla\bar{\boldsymbol{m}}(s)\right\|_2^2 \mathrm{d}s \leqslant C\int_0^t \left\|\bar{\boldsymbol{m}}(s)\right\|_2^2 \mathrm{d}s + C\int_0^t \left\|\bar{\boldsymbol{H}}(s)\right\|_2^2 \mathrm{d}s.$$

Note that the time variable t is not larger that T_0. Next we use Gronwall's lemma, and change generic constant C if necessary to obtain

$$\begin{aligned}
\left\|\bar{\boldsymbol{m}}(t)\right\|_2^2 &\leqslant C\int_0^t \left\|\bar{\boldsymbol{H}}(s)\right\|_2^2 \mathrm{d}s + \left\|\bar{\boldsymbol{m}}(0)\right\|_2^2 \\
&\quad + \int_0^t \left(C\int_0^s \left\|\bar{\boldsymbol{H}}(\tau)\right\|_2^2 \mathrm{d}\tau + \left\|\bar{\boldsymbol{m}}(0)\right\|_2^2\right) Ce^{(t-s)C}\,\mathrm{d}s \\
&\leqslant C\int_0^t \left\|\bar{\boldsymbol{H}}(s)\right\|_2^2 \mathrm{d}s + \left\|\bar{\boldsymbol{m}}(0)\right\|_2^2 \\
&\quad + \int_0^t \left(C\int_0^t \left\|\bar{\boldsymbol{H}}(\tau)\right\|_2^2 \mathrm{d}\tau + \left\|\bar{\boldsymbol{m}}(0)\right\|_2^2\right) Ce^{T_0C}\,\mathrm{d}s \\
&\leqslant C\int_0^t \left\|\bar{\boldsymbol{H}}(s)\right\|_2^2 \mathrm{d}s + (1+T_0C)\left\|\bar{\boldsymbol{m}}(0)\right\|_2^2 + T_0C\int_0^t \left\|\bar{\boldsymbol{H}}(\tau)\right\|_2^2 \mathrm{d}\tau \\
&\leqslant C\int_0^t \left\|\bar{\boldsymbol{H}}(s)\right\|_2^2 \mathrm{d}s + C\left\|\bar{\boldsymbol{m}}(0)\right\|_2^2. \qquad (25)
\end{aligned}$$

Direct use of Maxwell's equations to obtain estimate on the term $\|\bar{\boldsymbol{H}}\|_2$ would not be successful at this stage, because on the right-hand side of Maxwell's equations the term $\|\partial_t\bar{\boldsymbol{m}}\|_2$ arises and up to now we do not have any estimate on it. Therefore we need to obtain first estimate on $\|\Delta\bar{\boldsymbol{m}}\|_2$, which will be used to estimate $\|\partial_t\bar{\boldsymbol{m}}\|_2$.

Multiplication (24) by $-\Delta\bar{\boldsymbol{m}}$ and integration over Ω gives

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\nabla\bar{\boldsymbol{m}}\|_2^2 + \alpha\|\Delta\bar{\boldsymbol{m}}\|_2^2 \\
&\quad \leqslant \alpha\|\nabla\boldsymbol{m}_1\|_4^2\|\bar{\boldsymbol{m}}\|_{L^\infty}\|\Delta\bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\nabla\bar{\boldsymbol{m}}\|_4\big(\|\nabla\boldsymbol{m}_1\|_4 + \|\nabla\boldsymbol{m}_2\|_4\big)\|\boldsymbol{m}_2\|_{L^\infty}\|\Delta\bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\bar{\boldsymbol{m}}\|_{L^\infty}\|\Delta\boldsymbol{m}_1\|_2\|\Delta\bar{\boldsymbol{m}}\|_2 + \|\bar{\boldsymbol{m}}\|_{L^\infty}\|\boldsymbol{H}_1\|_2\|\Delta\bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\boldsymbol{m}_2\|_{L^\infty}\|\bar{\boldsymbol{H}}\|_2\|\Delta\bar{\boldsymbol{m}}\|_2 + \|\bar{\boldsymbol{m}}\|_{L^\infty}\|\boldsymbol{m}_1\|_{L^\infty}\|\boldsymbol{H}_1\|_2\|\Delta\bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\boldsymbol{m}_2\|_{L^\infty}\|\bar{\boldsymbol{m}}\|_{L^\infty}\|\boldsymbol{H}_1\|_2\|\Delta\bar{\boldsymbol{m}}\|_2 + \|\boldsymbol{m}_2\|_{L^\infty}^2\|\bar{\boldsymbol{H}}\|_2\|\Delta\bar{\boldsymbol{m}}\|_2.
\end{aligned}$$

From the embeddings $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and (19) we absorb the terms

$$\|\boldsymbol{m}_i\|_{L^\infty}, \quad \|\Delta \boldsymbol{m}_i\|_2, \quad \|\nabla \boldsymbol{m}_i\|_4, \quad \|\boldsymbol{H}_i\|_2$$

under a constant. In what remains we use the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, and the Young inequality to obtain

$$\partial_t \|\nabla \bar{\boldsymbol{m}}\|_2^2 + \|\Delta \bar{\boldsymbol{m}}\|_2^2 \leqslant C\|\bar{\boldsymbol{m}}\|_4 \|\Delta \bar{\boldsymbol{m}}\|_2 + C\|\nabla \bar{\boldsymbol{m}}\|_4 \|\Delta \bar{\boldsymbol{m}}\|_2 + C_\varepsilon \|\bar{\boldsymbol{H}}\|_2^2 + \varepsilon \|\Delta \bar{\boldsymbol{m}}\|_2^2.$$

Next we apply (10) and twice the Young inequality. First with exponents 8/7 and 8 and then with exponents equal to 2. We obtain

$$\begin{aligned}
\partial_t \|\nabla \bar{\boldsymbol{m}}\|_2^2 + \|\Delta \bar{\boldsymbol{m}}\|_2^2 &\leqslant C\|\nabla \bar{\boldsymbol{m}}\|_2^{1/4} \|\Delta \bar{\boldsymbol{m}}\|_2^{7/4} + C\|\nabla \bar{\boldsymbol{m}}\|_2 \|\Delta \bar{\boldsymbol{m}}\|_2 \\
&\quad + C\|\bar{\boldsymbol{m}}\|_2 \|\Delta \bar{\boldsymbol{m}}\|_2 + C_\varepsilon \|\bar{\boldsymbol{H}}\|_2^2 + \varepsilon \|\Delta \bar{\boldsymbol{m}}\|_2^2 \\
&\leqslant C_\varepsilon \|\bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon \|\nabla \bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon \|\bar{\boldsymbol{H}}\|_2^2 + \varepsilon \|\Delta \bar{\boldsymbol{m}}\|_2^2.
\end{aligned}$$

Integrating over time interval $\langle 0, t\rangle$, setting ε small enough, and using (25) and Gronwall's lemma we conclude

$$\left\|\nabla \bar{\boldsymbol{m}}(t)\right\|_2^2 + \int_0^t \left\|\Delta \bar{\boldsymbol{m}}(s)\right\|_2^2 \mathrm{d}s \leqslant C \int_0^t \left\|\bar{\boldsymbol{H}}(s)\right\|_2^2 \mathrm{d}s + C\left\|\nabla \bar{\boldsymbol{m}}(0)\right\|_2^2. \tag{26}$$

Further, we continue with estimating $\|\partial_t \bar{\boldsymbol{m}}\|_2$. Take (24), multiply it by $\partial_t \bar{\boldsymbol{m}}$ and integrate over Ω to get

$$\begin{aligned}
&\|\partial_t \bar{\boldsymbol{m}}\|_2^2 + \frac{1}{2}\partial_t \|\nabla \bar{\boldsymbol{m}}\|_2^2 \\
&\quad \leqslant \|\nabla \boldsymbol{m}_1\|_4^2 \|\bar{\boldsymbol{m}}\|_{L^\infty} \|\partial_t \bar{\boldsymbol{m}}\|_2 + \|\nabla \bar{\boldsymbol{m}}\|_4 \big(\|\nabla \boldsymbol{m}_1\|_4 + \|\nabla \boldsymbol{m}_2\|_4\big) \|\boldsymbol{m}_2\|_{L^\infty} \|\partial_t \bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\bar{\boldsymbol{m}}\|_{L^\infty} \|\Delta \boldsymbol{m}_1\|_2 \|\partial_t \bar{\boldsymbol{m}}\|_2 + \|\boldsymbol{m}_2\|_{L^\infty} \|\Delta \bar{\boldsymbol{m}}\|_2 \|\partial_t \bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\bar{\boldsymbol{m}}\|_{L^\infty} \|\boldsymbol{H}_1\|_2 \|\partial_t \bar{\boldsymbol{m}}\|_2 + \|\boldsymbol{m}_2\|_{L^\infty} \|\bar{\boldsymbol{H}}\|_2 \|\partial_t \bar{\boldsymbol{m}}\|_2 \\
&\qquad + \|\bar{\boldsymbol{m}}\|_{L^\infty} \|\boldsymbol{m}_1\|_{L^\infty} \|\boldsymbol{H}_1\|_2 \|\partial_t \bar{\boldsymbol{m}}\|_2 + \|\boldsymbol{m}_2\|_{L^\infty}^2 \|\bar{\boldsymbol{H}}\|_2 \|\partial_t \bar{\boldsymbol{m}}\|_2.
\end{aligned}$$

We can again absorb those terms containing $\boldsymbol{m}_i$ and $\boldsymbol{H}_i$ under a constant. Then we use the embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$, and thus

$$\|\partial_t \bar{\boldsymbol{m}}\|_2^2 + \frac{1}{2}\partial_t \|\nabla \bar{\boldsymbol{m}}\|_2^2 \leqslant C_\varepsilon \|\nabla \bar{\boldsymbol{m}}\|_4^2 + C_\varepsilon \|\Delta \bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon \|\bar{\boldsymbol{H}}\|_2^2 + \varepsilon \|\partial_t \bar{\boldsymbol{m}}\|_2^2.$$

Notice that we skipped the term $\|\bar{\boldsymbol{m}}\|_4$. However, this term can be treated on the same way as before, when deriving (26). Further, the embedding $W^{2,2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and setting ε small enough gives

$$\|\partial_t \bar{\boldsymbol{m}}\|_2^2 + \partial_t \|\nabla \bar{\boldsymbol{m}}\|_2^2 \leqslant C_\varepsilon \|\nabla \bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon \|\Delta \bar{\boldsymbol{m}}\|_2^2 + C_\varepsilon \|\bar{\boldsymbol{H}}\|_2^2.$$

Integration over time and the use of (26) leads to

$$\int_0^t \left\|\partial_t \bar{\boldsymbol{m}}(s)\right\|_2^2 \mathrm{d}s + \left\|\nabla \bar{\boldsymbol{m}}(t)\right\|_2^2 \leqslant C_\varepsilon \int_0^t \left\|\nabla \bar{\boldsymbol{m}}(s)\right\|_2^2 \mathrm{d}s + C_\varepsilon \int_0^t \|\bar{\boldsymbol{H}}\|_2^2 + \left\|\nabla \bar{\boldsymbol{m}}(0)\right\|_2^2.$$

Finally, we use Gronwall's lemma to arrive at

$$\int_0^t \|\partial_t \bar{\boldsymbol{m}}(s)\|_2^2 \, \mathrm{d}s + \|\nabla \bar{\boldsymbol{m}}(t)\|_2^2 \leqslant C_\varepsilon \int_0^t \|\bar{\boldsymbol{H}}\|_2^2 + C\|\nabla \bar{\boldsymbol{m}}(0)\|_2^2. \tag{27}$$

Now, we are ready to incorporate Maxwell's equations. Since (2) and (3) are linear we can directly consider both equations valid also for the triple $(\bar{\boldsymbol{m}}, \bar{\boldsymbol{E}}, \bar{\boldsymbol{H}})$

$$\partial_t \bar{\boldsymbol{E}} + \sigma \bar{\boldsymbol{E}} - \nabla \times \bar{\boldsymbol{H}} = 0, \qquad \partial_t \bar{\boldsymbol{H}} + \nabla \times \bar{\boldsymbol{E}} = -\beta \partial_t \bar{\boldsymbol{m}}.$$

Next, we multiply the previous equations by $\bar{\boldsymbol{E}}, \bar{\boldsymbol{H}}$, respectively. We integrate them over Ω and after summation, using the Young inequality, we obtain

$$\frac{1}{2}\partial_t \|\bar{\boldsymbol{E}}\|_2^2 + \frac{1}{2}\partial_t \|\bar{\boldsymbol{H}}\|_2^2 + \sigma \|\bar{\boldsymbol{E}}\|_2^2 \leqslant C\|\partial_t \bar{\boldsymbol{m}}\|_2^2 + C\|\bar{\boldsymbol{H}}\|_2^2.$$

Integrating over time interval and using (27) we get

$$\|\bar{\boldsymbol{E}}(t)\|_2^2 + \|\bar{\boldsymbol{H}}(t)\|_2^2 \leqslant C_\varepsilon \int_0^t \|\bar{\boldsymbol{H}}\|_2^2 + \|\bar{\boldsymbol{E}}(0)\|_2^2 + \|\bar{\boldsymbol{H}}(0)\|_2^2 + C\|\nabla \bar{\boldsymbol{m}}(0)\|_2^2,$$

which after using Gronwall's lemma gives

$$\|\bar{\boldsymbol{E}}(t)\|_2^2 + \|\bar{\boldsymbol{H}}(t)\|_2^2 \leqslant C\big(\|\bar{\boldsymbol{E}}(0)\|_2^2 + \|\bar{\boldsymbol{H}}(0)\|_2^2 + C\|\nabla \bar{\boldsymbol{m}}(0)\|_2^2\big).$$

Going back to the relations (25)–(27) we arrive at

$$\begin{aligned}
&\sup_{0<t<T_0} \big[\|\bar{\boldsymbol{m}}(t)\|_{W^{1,2}} + \|\bar{\boldsymbol{E}}(t)\|_2 + \|\bar{\boldsymbol{H}}(t)\|_2\big] + \int_0^{T_0} \big(\|\Delta \bar{\boldsymbol{m}}(s)\|_2^2 + \|\partial_t \bar{\boldsymbol{m}}(s)\|_2^2 \, \mathrm{d}s\big) \\
&\quad \leqslant C\big(\|\bar{\boldsymbol{E}}(0)\|_2^2 + \|\bar{\boldsymbol{H}}(0)\|_2^2 + C\|\bar{\boldsymbol{m}}(0)\|_{W^{1,2}}^2\big),
\end{aligned}$$

which finally concludes the proof of the theorem. □

6. Conclusions

In this work we derived higher regularity results for the space-periodic solution to the coupled Maxwell–Landau–Lifshitz system of ferromagnetism. This led us to the existence result stated in Theorem 3 and finally to the proof of uniqueness of the solution stated in Theorem 4. We have extended the previous results on existence of the solution

$$\begin{aligned}
&\boldsymbol{m}(x,t) \in L^\infty\big(0,T; W^{1,2}(\Omega)\big), \\
&\boldsymbol{H}(x,t), \boldsymbol{E}(x,t) \in L^\infty\big(0,T; L^2(\Omega)\big).
\end{aligned}$$

We stated existence and uniqueness of the solution with higher regularity

$$\begin{aligned}
&\boldsymbol{m}(x,t) \in L^\infty\big(0,T; W^{2,2}(\Omega)\big), \\
&\boldsymbol{H}(x,t), \boldsymbol{E}(x,t) \in L^\infty\big(0,T; W^{1,2}(\Omega)\big),
\end{aligned}$$

assuming higher regularity of initial conditions. We point out that it was necessary to use a different technique then the technique used in [12].

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